The Analysis of Large Deformation of Beam by Functionally Graded Material (FGM) Under Uniform Transverse Loading

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<td>In this paper, we study the analysis of the large beam deformation (FGM) under the uniform transverse loading. The mechanical properties of the beam including the modulus of elasticity and the Poisson coefficient are a function of the thickness of the beam (The Power Distribution Law). Principle equations for the FG-beams are obtained using the Von-Carmen theory for large deformities. The results are obtained by minimizing the total potential energy and solving it. The numerical examples are presented for this method. In this paper, the effect of material properties on the stress basin is examined from a thickness perspective. It discusses the effects of nonlinear terms in strain-relational relations.</td>
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1. Introduction

For the first time in 1980, Functionally Graded Materials were reported as advanced composite materials. At first, these materials used applications in both nuclear reactors and aerospace structures. Nevertheless, their basis was gradually increasing. Nowadays, they are used as components of structures working in very hot environments. Figure 1 shows how the phases are arranged in FG-materials. Phil Sang Lee and et al. [1] provided a general and three-dimensional analysis of the large elastoplastic deformation by finite element for a beam with a cross-section of L. In the paper, a numerical method is used to analyze the large elastoplastic deformation. S. Agarwal et al. [2] considered the analysis of the large deformation of non-homogeneous beams (composite and FGM) using a precision linear static method. Chuks N. Oguibe et al. [3] investigated the analysis of the large-scale deformation of a multi-layer-beam under impact loading. Banerjee et al. [4] investigated the great deformation of a recursive beam under a combination load. They consist of uniformly distributed load and a centralized load on the free-flowing head of the beam. The governing equation is extracted by using the shear force formulation instead of the formulation of the bending moment. BS Shuvartsman [5] presents the great deformation problem of a spring-headed recuperator beam under a focused follower force at the tip of the beam. In this paper, analysis of the large beam deformation is performed as a graded function (FGM). It is obtained by using the Van-Carmen theory for large dislocations under uniform transverse loading. The results are obtained by minimizing the total potential energy and solving it. S.A. Momeni et al. [6] presents a size-dependent formulation for the Euler-Bernoulli nano- and micro-beams made of functionally graded materials (FGMs). The formulation is developed on the basis of the second strain gradient theory (SSGT). This theory is a powerful non-classical continuum theory capable of capturing the small-scale effects in the mechanical behavior of small-scale structures. Ugurcan Eroglu [7] examines arbitrarily large in-plane deflections of planar curved beams made of Functionally Graded Materials (FGM). Geometrically exact beam theory is revisited, but the material properties are considered as an arbitrary function of the position on the cross-section of the beam to derive the governing differential equation system. Justin Murin and et al. [8] present a homogenized beam finite element for modal analysis considering a double symmetric cross-section made of a Functionally Graded Material (FGM). The material properties in a real beam can vary continuously in longitudinal direction while the variation with respect to the transversal and lateral directions is assumed to be symmetric in a continuous or discontinuous manner. Jun Lin and et al. [9] simulate geometrically nonlinear bending

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deformation of Functionally Graded Beams (FGBs) with variable thickness by a meshless Smoothed Hydrodynamic Particle (SPH) method. The material properties of FGB are assumed to be varied smoothly in the thickness according to exponent-law distribution.

Figure 1. The method of laying phases in the FG-material

2. The Theoretical Formulation

Large Deformation of Beam By Functionally Graded Materials (FGM) Under Uniform Transverse Loading theoretical formulation are presented here.

2.1. Properties of FG-materials

An FG of length and thickness “h” made from the combination of ceramics and metals are considered as shown in figure 2. It is assumed that the composition of the ceramic and the metal changes so that the upper surface of the beam is more than ceramic, and besides its lower surface is more metal. Hence, the material properties of the FG are like the modulus of elasticity (E) and the shear modulus (G). In addition, Poisson’s coefficient (\(\nu\)) is a function of the depth (z) of the beam. In this section, a simple power distribution law is used that includes all desired properties. The functional relationship between (\(\nu\)) and (E) is given in terms of (z) for the FG-beams of ceramic and metal. ("\(\xi\)"") specifies the property of the beam, and furthermore, the indexes 1 and 2 could indicate the upper and lower levels of the beam respectively [6].

\[
\xi(z) = \frac{\xi_2}{\xi_{12}} \left( \frac{z}{h} + \frac{1}{2} \right)^n + \xi_2, \quad \xi_{12} = \xi_1 - \xi_2
\]  

Figure 2. FG beam with simple supports and extensive transverse force

2.2. Ruling Equations

2.2.1. Equations governing the FG-beam

In order to write the governing equations for the FG array, the following assumptions are applied by taking into account classical theory.
1. The deformation of the beam is larger than its thickness is.

2. Both the modulus of elasticity and the Poisson coefficient of the FG-beam is a function of coordinates along its thickness.

### 2.2.2. Tension Field in FG-beam

According to figure 3, the displacement components for a common point in the x and z directions are as follows.

\[
\begin{align*}
    u(x, z) &= u_0(x) - z \frac{\partial w(x)}{\partial x} \\
    w(x, z) &= w_0(x)
\end{align*}
\]  

(2)

\[ u_0(x) \text{ and } w_0(x) \text{ are the components of the displacement at the middle of the beam.} \]

Using the classical theory of assumptions:

\[
\varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 - z \left( \frac{\partial^2 w}{\partial x^2} \right) \\ \frac{\partial w}{\partial z} \\ 2 \frac{\partial w}{\partial x} \end{bmatrix}
\]

(3)

The stress-strain relation in plate stress states for a general point of the beam plate is presented as follows:

\[
\sigma = \begin{bmatrix} Q_{11} & Q_{13} & 0 \\ Q_{31} & Q_{33} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{xz} \end{bmatrix}
\]

(4)

As we have:

\[
\begin{align*}
    Q_{11} &= Q_{33} = \frac{E(z)}{1 - \nu^2(z)} \\
    Q_{13} &= Q_{31} = \frac{\nu(z)E(z)}{1 - \nu^2(z)} \\
    Q_{66} &= \frac{E(z)}{2(1 + \nu(z))}
\end{align*}
\]

(5)

The rigidity of the beam is determined as follows.
Strain energy per unit volume,

\[ U_s = \frac{1}{2} \int [\varepsilon^T \{A\} \varepsilon - 2\varepsilon^T \{B\} \delta + \delta^T \{D\} \delta] \, dx 
\]

\[ = \frac{1}{2} \int [2\varepsilon_{uu}^T \{A\} \varepsilon_{uu} - 2\varepsilon_{uu}^T \{B\} \delta] 
\]

\[ + \frac{1}{2} \int [\varepsilon_{ww}^T \{A\} \varepsilon_{ww}] \, dx 
\]

That

\[ \varepsilon_{uu} = \begin{bmatrix} \frac{\partial w}{\partial x} \\
\frac{\partial w}{\partial z} \\
2^n \frac{\partial w}{\partial \lambda} \end{bmatrix}, \quad \varepsilon_{ww} = \begin{bmatrix} 0 \\
0 \\
0 \end{bmatrix}, \quad \delta = \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\
0 \\
0 \end{bmatrix} \]

The purpose of this paper is to evaluate beam responses under widespread uniform loads. Given that there is an external force in the problem, the total energy is equal to the sum of the strain energy. The energy of the external load potential is uniform. That’s:

\[ V = U_s + V_q \]

Potential energy load is uniform and equal to

\[ V_q = \int q \, w \, dx \]

Where \( q \) is a uniform load [10].

2.3. Fixed Propeller Conditions FG-beam

In this paper, it is assumed that the reinforcement of the beam is simple. There is no page displacement. That is, it prevents the edges from moving in the \( z \) direction. Therefore, we assume the displacement field according to figure 2 as follows:

\[ u = \sum_m k_m \sin \left( \frac{2m\pi x}{a} \right) \]

\[ w = \sum_m c_m \sin \left( \frac{m\pi x}{a} \right) \]

\[ k_m \text{ and } c_m \text{ coefficients are unknown displacement } u \text{ and } w \text{ are selected to estimate kinematic condition along the edges of the beam in a longitudinal direction.} \]

2.4. Analysis process

After obtaining the total potential energy of the sheet using relations (7), (9) and (10), we make a partial derivation of the total potential of the sheet relative to the unknown coefficients of \( k_m \) and \( c_m \). In this way, a nonlinear equation system is obtained based on \( k_m \) and \( c_m \). The number of nonlinear equations of this machine depends on the number of terms used (the value of the counter \( m \)) in equations (11). By solving this system, the equations are solved by one numerical method (here they are solved by Newton-Raphson method). The values of \( k_m \) and \( c_m \) are obtained by inserting these values into relations (11), and then we can obtain the equations of displacements at any point in the arc. Then, using (3) and (4), we can obtain the values of strains and stresses in different points of the beam, respectively.
3. Results and Discussion

In this section, we obtain the displacement and stresses for an FG-beam with simple retention conditions using the proposed method. We assume that the properties changes along the beam thickness follow the power distribution function. This function was represented by Equation (1). As you can see, this function has a variable parameter called n. It is in power in the equation. The changes in (n) make the composition of the FG-beam change in line with the thickness in terms of the volume fraction of its constituent material. For instance, for n = 0 of the beam is homogeneous, and it is made of ceramic. For n = infinity, the beam will be homogeneous, and it is made of metal. The ceramics used in this paper are alumina ($Al_2O_3$) with a modulus of elasticity of 380 G.Pascal and a Poisson coefficient. The metal used in it is aluminum with a modulus of elasticity of 70 G. Pascal and a Poisson coefficient 0.33. For convenience, the two non-dimensional parameters are defined as follows.

No-dimensional displacement $W = \frac{w}{h}$

Loading with no dimension $Q = \frac{qa^4}{E_{Al}h^4}$

In this formula, $E_{Al}$ is the modulus of elasticity of the aluminum. The variations in the ceramic volume fraction in the thickness of the beam are shown in figure 1 for different values of n. In figure 2, the dimensionless displacement of the center of the FG-beam is represented by different values of n for different values of the dimensionless load Q. As shown in this diagram, increasing the value of the parameter (n) increases the dimensionless displacement of the center of the beam. This is due to the decrease in the modulus of elasticity of the beam in its various layers in thickness. As the parameter (n) increases from zero to infinite, and it gradually decreases the number of ceramic layers of the FG-beam. Instead, they increase their metal content. It is also observed that the displacement of the beam was not linear in terms of its load.

This is due to the presence of nonlinear semiconductors in the strain-displacement relation. This reduces the amount of displacement. It is worth noting that the effects of nonlinear terms are more pronounced in the higher values of the parameter (n) [11].

![Figure 4. Characteristics of the volume fraction of ceramic changes in the width of the FG-beam for different values of (n)](image)

In figure 4, the curves of the stress values in the center of the FG-beam are plotted for different values of the (n) parameter, in the case of a subsequent load value of one.

As we know, in linear mode, the stress in the width of the homogeneous beam does not depend on the modulus of elasticity of the beam. However, according to this diagram, we observe that in a nonlinear state (large deformations) the stress values depend on the value of the modulus of elasticity of the beam. This result is obtained by comparing two curves for n = 0 and n = infinity. For both two modes, the beam is homogeneous, and they are respectively made of ceramic and metal. In addition, as shown in this graph, stress variations for different values of the parameter (n) are not linear. The reason for this is the continuous change of properties within the FG-beam width. Due to the modulus of elasticity varies across the thickness of the beam. It depends on the elastic modulus in the nonlinear analysis of stress. Therefore, the stress variation graph for values other than zero and infinite (ns) would not be linear. The maximum
variation of tension for a mode is $n = 0.5$. This is due to the sudden variation of the modulus of elasticity in the width of the beam for values smaller than one parameter ($n$).

**Figure 5.** The dimensionless displacement of the center of the FG-beam with different values of ($n$) and for different load sizes without dimension $Q$

**Figure 6.** The stress values at the center of the beam for different values of the parameter $n$ ($Q = -1$)

4. **Conclusions**

In this paper, we have investigated the analysis of the large beam deformation as a functionally graded (FGM) under uniform transverse loading. The mechanical properties of the beam including the modulus of elasticity and the Poisson coefficient are the function of the beam thickness (power distribution law). Principle equations for FG-beams were obtained using Van-Carmen's theory for large dislocations. The results were obtained by minimizing total potential energy and solving it. The velocity diagrams of the center of the beam and the tensile graphs were obtained in the width of the beam for various loading. Different values of ($n$) (in relation 1) were investigated for the distribution of properties along the width of the beam. Comparison of the results using Tymoshenko's relations took place for $n = 0$ (homogeneous ceramic beam) and for $n = \infty$ (homogeneous aluminum). This indicates the accuracy of the results obtained with this method for other scenarios. In addition, the effects of nonlinear terms in the existing relationships were discussed.
References


